ON EFFECTIVE LOG IITAKA FIBRATION FOR 3-FOLDS AND 4-FOLDS

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ABSTRACT. We prove the effectiveness of the log Iitaka fibration in Kodaira codimension two for varieties of dimension$\leq$ 4. In particular, we finish the proof of effective log Iitaka fibration in dimension two. Also, we show that for the log Iitaka fibration, if the fiber is of dimension two, the denominator of the moduli part is bounded.

1. INTRODUCTION

One of the main problems in birational algebraic geometry is to understand the structure of pluricanonical maps. If X is a smooth complex projective variety of dimension n, then we define the pluricanonical maps

$$\phi_{rK_X} : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{O}_X(rK_X)))$$

determined by the linear system $|rK_X|$. If this linear system is nonempty for some natural number r, which is conjectually equivalent to X being non-uniruled, then for r sufficiently divisible the maps ϕ_{rK_X} become birational to a fixed algebraic fiber space $\phi : X' \longrightarrow Y'$ called the Iitaka fibration of X and the Kodaira dimension $\kappa(X)$ is set to be $\dim Y'$. It is then natural to look for a uniform r (that is an integer r which depends only on the dimension of X) for which we are inducing a map birational to the Iitaka fibration. After the monumental work [BCHM06] of proving the finite generation of the pluricanonical rings, the effective Iitaka fibration problem is largely related to finding generators which are in less than a uniform degree for a given dimension. If $\kappa(X) = \dim X$, then X is called of general type and in this case Hacon-McKernan [HM06] and Takayama [Ta06] following ideas of Tsuji, have shown that for a smooth projective variety of general type and dimension n there exists an integer r_n, which depends only on n, such that ϕ_{rK_X} is birational for $r \geq r_n$. The question remains widely open when the Kodaira dimension in not maximal and it is completely known only up to dimension three (cf. [FM00, VZ07, Ri07]).

The current approach for studying this problem is to use Kawamata's canonical bundle formula which identifies the pluricanonical ring of X with the pluricanonical ring of a pair $(Y, \Delta + L)$ of log general type (cf. [FM00], [PS09]), where (Y, Δ) is a *klt* pair and L is a $\mathbb{Q}$-line bundle coming from variation of Hodge structure. This raises the natural question of a log analogue of the above statement. Another setting where boundary divisors naturally occur is in moduli problems. Because

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of the presence of a boundary divisor the best statement that we can hope for is the following: for (X, D) a *klt* pair of Kodaira dimension $\kappa \geq 0$ there is a natural number m, that depends only on the coefficients of Δ and the dimension of X such that $||\lfloor m(K_X + \Delta) \rfloor||$ is birational to the Iitaka fibration. To be able to use the canonical bundle formula inductively we have to allow that the coefficients of Δ lie in a possibly infinite set of rational numbers. It turns out that the natural assumption on the set of coefficients is the *descending chain condition* (DCC). At the first glance, it may seem to be quite technical and somewhat artificial. However, this is a very natural and useful condition arising in many problems. The interested reader is recommended to read [Sh92], [Ko94] for more detailed discussions of this. Thus the general conjecture is formulated as follows:

1.1. **Conjecture.** *(Effective Log Iitaka Fibration Conjecture) Let (X, Δ) be a klt (or $\epsilon - lc$) pair such that $K_X + \Delta$ is pseudo-effective and the coefficients in Δ are in a DCC set $\mathcal{A}$. Then there is a constant r depending only on the dimension X and on $\mathcal{A}$ such that $||\lfloor r(K_X + \Delta) \rfloor||$ induces a birational map.*

In this paper we prove the following:

1.2. **Theorem.** *Let (X, Δ) be a klt pair of dimension two, three or four and Kodaira codimension two. Assume that the coefficients of Δ are in a DCC set of rational numbers $\mathcal{A} \subset [0, 1]$. Then there is an explicitly computable constant m depending only on the set $\mathcal{A}$ such that $\lfloor m(K_X + \Delta) \rfloor$ induces the Iitaka fibration.*

In particular, this completes the effectiveness of log Iitaka fibration in dimension two (for the other cases, see [To08]). The dimension two case relies on results by Alexeev and Alexeev-Mori [Al94, AM04].

A key tool in the dimension three and the dimension four case is the canonical bundle formula (cf. [Ka98], [FM00], [Ko07]). In our case it roughly says that if $f : X \longrightarrow Y$ is the Iitaka fibration for $K_X + \Delta$ and $K_X + \Delta = f^*D$ for some $\mathbb{Q}$-divisor D on Y then we can define the *discriminant* or *divisorial part* on Y for $K_X + \Delta$ to be the $\mathbb{Q}$-Weil divisor $B_Y := \sum_P b_P P$, where $1 - b_P$ is the maximal real number t such that the log pair $(X, \Delta + tf^*(P))$ has log canonical singularities over the generic point of P. The sum runs over all codimension one points of Y, but it has finite support. The *moduli part* or *J-part* is the $\mathbb{Q}$-line bundle M_Y on Y which satisfies

$$K_X + \Delta = f^*(K_Y + B_Y + M_Y).$$

If F is the general fiber of f and if $h^0(F, b(K_F + \Delta_{|F})) \neq 0$, then for every integer r divisible by b we have

$$H^0(X, \lfloor r(K_X + \Delta) \rfloor) = H^0(Y, \lfloor r(K_Y + B_Y + M_Y) \rfloor).$$

Using ideas of Mori and Fujino, and the dimension two case of Theorem 1.2, we prove an analogue of Mori and Fujino bounding the denominators in relative dimension two.

1.3. **Theorem.** *When the relative dimension of f is two, there is a natural number m depending only on the coefficients of Δ such that mM is Cartier.*

In the dimension three case of Theorem 1.2 the base is a curve and the conclusions follows (1.3) by standard arguments as in [FM00]. When the dimension is four a key observation is that many results in [AM04] can be improved by adding a nef divisor (4.4). Similar statements were essentially proved by Viehweg-Zhang [VZ07], but we will simplify their proof by putting it in the context of [Al94], [AM04] and the recent main stream investigations on adjoint linear systems. We should also remark that in ([PS09], Conjecture 7.13), Prokhorov-Shokurov list a series of conjectures concerning the solution of the effective Iitaka fibration problem. Putting our work in the frame of their conjectures, we indeed show Conjecture 7.13(2) in the relative dimension two case. However, we do not prove the semi-ampleness statements of Conjecture 7.13 (1) and (3).

The paper is structured as follow. In Section 2, we show that the DCC assumption on the coefficients of the boundary indeed forces that the *klt* surface pairs of log Kodaira dimension zero to be ϵ-log canonical, for some ϵ depending on the set of coefficients. In Section 3, we prove that under the same assumption there is a uniform b such that $b(K_S + B) \sim 0$. In Section 4, using ideas of Mori and Fujino, we deduce Theorem (1.3). And then we complete the proof of Theorem (1.2).

1.1. **Notations and Conventions.** We will work over the field of complex numbers $\mathbb{C}$. A $\mathbb{Q}$-Cartier divisor D is nef if $D \cdot C \geq 0$ for any curve C on X. We call two $\mathbb{Q}$-divisors D_1, D_2 $\mathbb{Q}$-linearly equivalent $D_1 \sim_{\mathbb{Q}} D_2$ if there exists an integer $m > 0$ such that mD_i are integral and linearly equivalent. We call two $\mathbb{Q}$-Cartier divisors D_1, D_2 numerically equivalent $D_1 \equiv D_2$ if $(D_1 - D_2) \cdot C = 0$ for any curve C on X. A log pair (X, Δ) is a normal variety X and an effective $\mathbb{Q}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A projective morphism $\mu : Y \longrightarrow X$ is a log resolution of the pair (X, Δ) if Y is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossing support. For such μ we write $\mu^*(K_X + \Delta) = K_Y + \Gamma$, and $\Gamma = \Sigma a_i \Gamma_i$ where Γ_i are distinct integral divisors. A pair is called *klt* (resp. *lc*, ϵ-*lc*) if there is a log resolution $\mu : Y \longrightarrow X$ such that in the above notation we have $a_i < 1$ (resp. $a_i \leq 1, a_i \leq 1 - \epsilon$). The number $1 - a_i$ is called log discrepancy of Γ_i with respect to the pair (X, Δ).

For conventions and results about DCC set we refer to [AM04], in particular, ([AM04], 2.2-2.7 and 3.4-3.6). When we say some quantity is bounded, it always

means there is a computable bound depending on the data we give. We will not keep tracking of the explicit bound, however, this does not require much effort by following the arguments in [AM04]. Most of the results about surfaces that we use appear already in [Al94], but to make the constants explicitly computable we refer to the more recent [AM04].

2. ϵ-LOG CANONICITY

The main result of this section is to show the following

2.1. Theorem. *Let $(S, B = \sum b_i B_i)$ be a klt projective surface pair. If $K_S + B \equiv 0$ and the coefficients of B are in a DCC set $\mathcal{A}$, then there exists an $\epsilon = \epsilon(\mathcal{A}) > 0$, which only depends on $\mathcal{A}$ such that (S, B) is ϵ-log canonical.*

Proof. We can run a minimal model program for S,

$$S = S_0 \to S_1 \to \cdots \to S_n.$$

By pushing forward B to S_n, we see that it suffices to prove the statement for S_n. So we need only to prove for the case that S is a log del Pezzo surface of Picard number 1, S admits a Fano contraction to some curve with a general fiber $\mathbb{P}^1$ or S has $K_S \equiv 0$. This is done in (2.4), (2.5) and (2.6). □

The main tool we use is the following theorem,

2.2. Theorem. [AM04, Theorem 3.2] *Let $\mathcal{A} \subset [0, 1]$ be an arbitrary DCC set. Then there exists a constant $\delta = \beta(\mathcal{A})$ depending only on $\mathcal{A}$ so that the following holds. Let X be a normal projective surface, B_j be divisors on X, and let b_j, x_j be positive real numbers. Assume that*

(1) *X is a singular $\mathbb{Q}$-factorial surface with $-K_X$ ample and $\rho(X) = 1$,*
(2) *$b_j > 0$ and $b_j \in \mathcal{A}$,*
(3) *$1 - \delta < x_j \leq 1$,*
(4) *at least one x_j is strictly less than 1,*
(5) *the pair $(X, \sum x_j b_j B_j)$ is lc.*

Then the divisor $K_X + \sum x_j b_j B_j$ is not numerical equivalent to 0.

Also we need the following lemma on DCC sets,

2.3. Lemma. [AM04, Lemma 3.6] *Let $\mathcal{A}$ be a DCC set containing 1. Let $a, \delta > 0$ such that $\delta \leq mf_2(\mathcal{A}, a)$ (see [AM04] (3.5) for the definition of the function $mf_2(\mathcal{A}, a)$). Then for any finite number of arbitrary b_j, $x_j \in R$ such that $0 < b_j \in \mathcal{A}$ and $1 - \delta < x_j \leq 1$ for all j, $x_j < 1$ for some j, then we have $\sum x_j b_j \neq a$.*

2.4. Lemma. *If S is a log del Pezzo surface of Picard number 1. $B = \sum b_j B_j$ is a $\mathbb{Q}$-divisor such that $K_S + B \equiv 0$, (S, B) is klt and the coefficients of B are in a DCC set $\mathcal{A}$. Then S is $\epsilon = \epsilon(\mathcal{A})$-log canonical.*

Proof. We assume $1 \in \mathcal{A}$. If (S, B) is not ϵ-log canonical, then we can extract a divisor $f : S' \to S$ such that

$$f^*(K_S + \sum b_j B_j) \equiv K_{S'} + \sum b_j B_j + (1-a)E,$$

with $a < \epsilon$. S' has Picard number 2, so it has two extremal rays, one of which is given by E. Contracting the other extremal ray we obtain a morphism $S' \to S_0$ to a surface S_0. In fact if S_0 were a curve, by taking the intersection of $K_{S'} + \sum b_j B_j + (1-a)E$ with a general fiber, we have

$$\sum n_i b_j + n(1-a) = 2,$$

where $n_i \geq 0$ and $n > 0$ are integers, which contradicts (2.3), provided we choose $\epsilon = m f_2(\mathcal{A}, 2)$. Then S_0 is a log del Pezzo of Picard number 1. Now g does not contract E, and

$$g_*(K_{S'} + \sum b_j B_j + (1-a)E) = K_{S_0} + (1-a)E + \sum b_j g_*(B_j) \equiv 0.$$

Since $K_S + B \equiv 0$ we still have that $(S_0, (1-a)E + \sum b_j g_*(B_j))$ is a *klt* pair. Then Theorem 2.2 applied to S_0 with $b_E = 1, x_E = (1-a)$, all other $x_j = 1$ and $\epsilon = \delta$ implies that $K_{S_0} + (1-a)E + \sum b_j g_*(B_j)$ is not numerically equivalent to zero which is a contradiction. $\qquad\square$

2.5. Lemma. *Let $(S, B) \to C$ be a morphism from a surface to a curve, such that the generic fiber is $\mathbb{P}^1$ and let (S, B) be klt with the coefficients of B in a DCC set $\mathcal{A}$. Then S is $\epsilon = \epsilon(\mathcal{A})$-log canonical.*

Proof. Suppose that this is not the case. Then extract a divisor $f : S' \to S$ such that

$$f^*(K_S + \sum b_j B_j) \equiv K_{S'} + \sum b_j B_j + (1-a)E,$$

with $a < \epsilon$. Running a minimal model program for $(S', \sum b_j B_j)$ that does not contracts E, we end with a surface S'' that is either log del Pezzo or admits a Fano contraction to a curve. Furthermore the coefficients of E in the pair (S'', B'') is $(1-a)$ and $S'' + B'' \equiv 0$.

In the case S'' is log del Pezzo, we get a contradiction as in (2.4). If S'' admits a Fano contraction to a curve, by intersecting with a general fiber, we can apply the argument in (2.4) again.

$\qquad\square$

Finally we deal with the case in which K_S is numerically trivial.

2.6. Lemma. *There is $\epsilon > 0$ such that every klt surface with $K_S \equiv 0$ is ϵ-log canonical.*

Proof. Set $\epsilon = \min\{\beta(\{1\}), m f_2(\{1\}', 2)\}$. Suppose that S is not ϵ-log canonical and extract a divisor E by $f : S' \to S$ such that

$$f^*(K_S) = K_{S'} + (1-a)E \text{ and } 0 < a < \epsilon.$$

Then running a minimal model for S', which does not contract E, we can apply one of the above two cases.

$\square$

3. Bounding the index

Observe that on the varying a, b and c the weighted projective spaces $\mathbb{P}(a, b, c)$ with three lines $(x_i = 0)$ forms an unbounded family of *lc* surface pairs with coefficients are in the DCC set $\{1\}$. However by restricting to the *klt* case we have the following boundedness result:

3.1. **Theorem.** *For (S, B) a klt pair such that $K_S + B \equiv 0$ and the coefficients of B are in a DCC set $\mathcal{A}$, there is a natural number $b = b(\mathcal{A})$, such that $b(K_S + B)$ is Cartier and $H^0(S, b(K_S + B)) = 1$.*

Form Theorem (2.1) we know that there is an $\epsilon > 0$ such that (S, B) is ϵ-log canonical and we will assume $0 < \epsilon < 1/\sqrt{3}$.

3.2. **Lemma.** [AM04, Lemma 1.2, Theorem 1.8] *Let X be a non singular projective surface and $B = \sum b_j B_j$ be an $\mathbb{R}$-divisor on X with $0 \le b_j \le 1 - \epsilon < 1$. Assume $K_X + B \equiv 0$, then*
   (1) *If E is an irreducible curve on X and $E^2 < 0$ then $E \cong \mathbb{P}^1$ and $E^2 > -2/\epsilon$; and*
   (2) *$\rho(X) \le 128/\epsilon^5$.*

3.3. **Proposition.** *(S, B) as in (3.1). There exists an integer $t = t(\epsilon)$ such that for any Weil divisor D on S, tD is Cartier.*

Proof. The argument is the parallel to the one given in ([AM04], Lemma 3.7), though there S is assumed to be a del Pezzo surface of Picard number 1.

Take the minimal resolution $f : \widetilde{S} \to S$ of S and write

$$f^*(K_S) = K_{\widetilde{S}} + \sum a_i E_i.$$

Applying (3.2), we conclude that the determinant t of the intersection matrix of the exceptional curves $(-F_i F_k)_{ik}$ is bounded by

$$t \le [2/\epsilon]^{[128/\epsilon^5]}.$$

Since S has rational singularities, for any Weil divisor D on S, tD is a Cartier divisor. $\square$

In the next Proposition we repeatedly use the trivial property of DCC set stated in the following Lemma.

3.4. **Lemma.** *Let c be a positive real number and $\mathcal{A} \subset [0, 1]$ a DCC set. Then there are a finite number of ways to write c as the sum of $a_i \in \mathcal{A}$.*

3.5. **Proposition.** *(S, B) as in (3.1). There exists an integer $t = t(\epsilon)$ such that $t(K_S + B)$ is Cartier.*

Proof. Running a minimal model program for S, and we end with a *klt* surface $f : S \to S'$. If $t(K'_S + B')$ is Cartier, since

$$f^*(t(K_{S'} + B')) \sim t(K_S + f_*^{-1}B' + a_iE_i)$$

and

$$K_S + f_*^{-1}B' + a_iE_i \equiv K_S + B \equiv 0,$$

we know $K_S + f_*^{-1}B' + a_iE_i = K_S + B$, thus it suffices to prove for (S', B'). S' is either of Kodaira dimension 0, has a contraction to a curve with general fiber $\mathbb{P}^1$ or a log del Pezzo surface, so we only need to prove the proposition in these cases.

(Case 1): S has Kodaira dimension 0. Then $B = 0$ and it follows directly from (3.3);

(Case 2): S admits a contraction to a curve $f : S \to C$ with general fiber $\mathbb{P}^1$. By taking the intersection of (S, B) with the general fiber, we have

$$2 = \sum_{f(B_j)\neq pt} a_j, \text{ where } a_j \in \mathcal{A}'.$$

On the other hand, applying the canonical bundle formula in this simple case, we have

$$K_S + B \equiv f^*(K_C + B_C) \equiv 0, \text{ where } B_C = \sum_{f(B_j)=P_j} \frac{a_j + n - 1}{n}P_j$$

From Lemma (3.4) we know that there are only finitely many choices of a_j, and we choose a natural number $N = N(\mathcal{A})$ that clears all the denominators of a_j.

(Case 3): S is a log del Pezzo surface of Picard number one. For t as in (3.3), $K_S + \sum b_jB_j \equiv 0$ implies that

$$-tK_S^2 = \sum tb_jB_j \cdot K_S.$$

Here tK_S^2 and $tB_j \cdot K_S$ are integers. From the proof of ([AM04], Lemma 3.7), we can bound tK_S^2 by

$$tK_S^2 \leq [2/\epsilon]^{[128/\epsilon^5]} \times ([2/\epsilon] + 2)^2.$$

Lemma 3.4 implies that there are only finitely many possible b_i appearing as the coefficients of the boundary divisor. So there is a positive number $N = N(\mathcal{A})$ such that NK_S and $N(b_jB_j)$ are all Cartier divisors.

$\square$

Because of the above discussion, if we take the minimal resolution $\pi : \widetilde{S} \to S$ of S, pull back $K_S + B$, then the denominators can be killed by a uniform multiple $t = t(\epsilon)$. Thus by running a minimal model program for $\widetilde{S}$ we only need to prove (3.1) in the case that S is a smooth minimal surface and that the coefficients of B all have the form r/t for some uniform t. When $K_S \equiv 0$, and hence $B = 0$ the result follows from the classification theory of smooth surfaces; when S is

rational, a Cartier divisor is numerically trivial if and only if it is a trivial divisor. Thus the only tricky case is dealt with in the next Lemma.

3.6. **Lemma.** *If S is a smooth minimal ruled surface over a curve of positive genus and there is a $\mathbb{Q}$-divisor B such that (S, B) is klt and $K_S + B \equiv 0$, then $S = E \times \mathbb{P}^1$, where E is an elliptic curve. In particular, if for some integer N, NB is an integer divisor, then $\mathcal{O}(N(K_S + B)) \cong \mathcal{O}_S$.*

Proof. We use conventions for ruled surfaces as in ([Ha77], V.2). Let $\pi : S \to C$ be $\mathbb{P}(\mathcal{E})$ over a curve C of positive genus. We can assume $H^0(\mathcal{E}) \neq 0$ and $H^0(\mathcal{E} \otimes L) = 0$ for any line bundle L of degree 1. Then $e = -\deg(\mathcal{E})$ is an invariant of S. There is a section C_0 such that $C_0|_{C_0} = \bigwedge^2 \mathcal{E}$, which we denote as D. Then $\deg(D) = -e$ and

$$K_S \sim -2C_0 + \pi^*(K_C + D).$$

Then we have,

$$0 = (K_S + B) \cdot C_0 = e + 2g - 2 + tC_0^2 + \sum_{B_i \neq C_0} b_i B_i \cdot C_0 \geq 2g - 2 + (1 - t)e,$$

where $t < 1$ is the coefficient of C_0 in B. So $e \leq 0$, and if $e = 0$, then $g = 1$. On the other hand if $e < 0$, ([Ha77], V.2.21) shows that

$$B \cdot C_0 \geq (2C_0 + eF) \cdot C_0 \geq -e,$$

so again we conclude $g = 1$. And the equality holds only when for each irreducible component B_i of $\mathrm{Supp}(B)$, $g_i : B_i \to C$ is unramified.

Now we have $B_i \equiv a(2C_0 + eF)$ and $B_i^2 = 0$.

Claim: For $g : B_i \to C$, there are line bundles M and N on B_i such that

$$0 \to \mathcal{O} \to g^*(\mathcal{E}) \otimes M \to N \to 0, \text{ and } \deg N = 0.$$

Proof of the claim: it suffices to show that for the induced section $h : B_i \to \mathbb{P}_{B_i}(g_i^* \mathcal{E})$ we have $h(B_i)^2 = 0$. But for $B_i \subset \mathbb{P}(\mathcal{E})$,

$$0 \to I_{B_i}/I_{B_i}^2 \to \Omega^1_{\mathbb{P}(\mathcal{E})} \otimes B_i \to \Omega^1_{B_i} \to 0,$$

where $\deg(I_{B_i}/I_{B_i}^2) = 0$. Since $B_i \to C$ is unramified, the same exact sequence also computes the conormal bundle of $h(B_i) \subset \mathbb{P}_{B_i}(g_i^* \mathcal{E})$ and hence we conclude that $h(B_i)^2 = 0$.

If $e > 0$, then $\mathcal{E}$ is indecomposable which also means it is stable. But the above exact sequence shows that $g^*(E)$ is not semi-stable, which is a contradiction.

When $e = 0$, if $\mathcal{E}$ is indecomposable, then E is given by a non-split extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0.$$

For any isogeny $g : B \to C$, if there is a surjection $g^*(\mathcal{E}) \to F \to 0$ for a line bundle F of degree 0, then it is given by the pull-back of the above exact sequence. So $B = 2C_0$, which contradicts to the assumption that (S, B) is *klt*. Similarly, if

$\mathcal{E} = \mathcal{O} \oplus L$ for some degree 0 bundle L, then we have at most two candidates for B_i, which again contradicts (S, B) being *klt*.

Finally, when $\mathcal{E} = \mathcal{O}^2$, then all candidates of B_i are proportional to C_0. Thus the last statement holds. $\qquad\square$

4. Effective Iitaka Fibration

In this section we prove the effectiveness of the log Iitaka fibration for (X, Δ) a *klt* pair of dimension three and four and log Kodaira codimension two.

Consider a log resolution $\pi : X' \longrightarrow X$ of (X, Δ) and write

$$\pi^*(K_X + \Delta) \equiv K_{X'} + (\pi^{-1})_*\Delta + \sum_i e_i E_i$$

with E_i exceptional. There is a natural number n such that $e_i < 1 - \frac{1}{n}$ for every i. Define

$$\Delta' = (\pi^{-1})_*\Delta + \sum(1 - \frac{1}{n})E_i.$$

Since all E_i are exceptional divisors, we have

$$H^0(X', \lfloor m(K_{X'} + \Delta')\rfloor) = H^0(X, \lfloor m(K_X + \Delta)\rfloor).$$

By replacing the $\mathcal{A}$ with the DCC set $\mathcal{A} \cup \{1 - \frac{1}{n} | n \in \mathbb{N}\}$, we can assume that X is smooth and Δ is simple normal crossing. Furthermore, we can assume that there is a morphism $f : X \longrightarrow C$ giving the Iitaka fibration for $K_X + \Delta$ with C a smooth projective variety of dimension m (cf. [La04], 2.1.C). For F the general fiber of f, we have that $\kappa(K_F + \Delta_F) = 0$.

From Kawamata's canonical bundle formula, we know that there are a $\mathbb{Q}$-line bundle $M = M^{SS}_{(M,\Delta)/C}$, a $\mathbb{Q}$-divisor B on C and a $\mathbb{Q}$-divisor R on X such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_C + M + B) + R,$$

where the terms in the formula satisfy the property that if $b(K_F + \Delta_F)$ is Cartier and $h^0(F, b(K_F + \Delta_F)) = 1$, then for any $r \in \mathbb{Z}_{\geq 0}$ divided by b,

$$H^0(X, \lfloor r(K_X + \Delta)\rfloor) = H^0(C, \lfloor r(K_C + M + B)\rfloor).$$

(cf. ([FM00], Theorem 4.5)). M and B are usually referred as the moduli part and the boundary part of the algebraic fiber space.

Proof of (1.3): First we can assume that (F, Δ_F) is a minimal pair that is $K_F + \Delta_F \equiv 0$. In fact, from ([Ko07], Definition 8.4.6), we know the moduli part M only depends on the birational class of the morphism $X \to C$, and we can choose one such that the generic fiber is minimal. Hence Theorem (2.1) implies that (F, Δ_F) is $\epsilon = \epsilon(\mathcal{A})$-lc.

Let d be the smallest positive integer such that $d(K_F + \Delta_F)$ is Cartier and $h^0(F, d(K_F + \Delta_F)) = 1$. Then we can construct the corresponding cyclic cover $g : E \to F$, and let $\phi : E' \to E$ be the minimal resolution of E. Then as in ([FM00], Theorem 3.1), it suffices to bound the second Betti number of E'.

Write $g^*(K_F + \Delta_F) = K_E + \Delta_E$. From ([KM98], 5.20.3), we know (E, Δ_E) is ϵ-log canonical, provided that so is (F, Δ_F), which is the case because of Theorem (2.1). Pulling back to E', we conclude that $(E', \Delta_{E'})$ is ϵ-log canonical. Then ([AM04], Theorem 1.8) implies that $\rho(E') \leq 128/\epsilon^5$. Thus the second Betti number of E' is bounded by a constant depending only on $\mathcal{A}$.

The remaining part of the proof was given in [To08, Section 3]. We give a sketch here. First we assume that the generic fiber (F, Δ_F) is log smooth, then as in [Ko07, 8.4.5(7)], we can define a local system $\mathbb{V}$ on an open set $U \subset X$. Furthermore, if the the monodromy action on $R^2 f_*(\mathbb{V})$ is unipotent, then the moduli part M is identified with the bottom Hodge filtration of $R^2 f_*(\mathbb{V})$, which is an integral divisor. In general, we take a log resolution $\widetilde{F} \to F$ and write

$$\pi^*(K_F + \Delta_F) = K_{\widetilde{F}} + \Delta_{\widetilde{F}} - G,$$

where $\lfloor \Delta_{\widetilde{F}} \rfloor$, $G \geq 0$ is an integral divisor and $\mathrm{Supp}(\Delta_{\widetilde{F}})$ has no common component with G. Near every codimension one point P, there is a Galois cover Y' of Y such that when we pull back everything to Y', the monodromy action becomes unipotent near P. Thus as in [FM00, 3.6], the remaining question is to bound the character of $G \to \mathbb{C}^*$ corresponding to the representation of G on $H^0(\widetilde{F}, \mathcal{O}(G))$. If we write $\widetilde{E}$ as the degree-b cyclic cover corresponding to $b(K_{\widetilde{F}} + \Delta_{\widetilde{F}} - G) \sim 0$, then $H^0(\widetilde{F}, \mathcal{O}(G))$ is a direct summand of the bottom piece of the Hodge filtration of $H^2(\widetilde{E}, \mathbb{C})$, which is also the bottom piece of $H^2(E', \mathbb{C})$ since $\widetilde{E}$ and E' are birational.

Then from [FM00, 3.8] we conclude that if the index of (F, Δ_F) is bounded by $b = b(\mathcal{A})$ and the second Betti number of the cyclic cover is bounded by $B = B(\mathcal{A})$, then the denominator of the moduli part is bounded by

$$a = b \cdot \mathrm{lcm}\{m \in \mathbb{Z}_{>0} | \varphi(m) \leq B\},$$

where φ denotes the Euler φ-function.

4.1. **Theorem.** *Notation as above. Assume that the coefficients of Δ are in a DCC set of rational numbers $\mathcal{A} \subset [0, 1]$. Let the dimension of X be either three or four and the dimension of the general fiber be two. Then there is an constant N depending only on the set $\mathcal{A}$ such that $\lfloor N(K_X + \Delta) \rfloor$ induces the Iitaka fibration.*

We prove the statement when the dimension of X is four, and leave the easier dimension three case to the reader. In fact, this follows directly from the argument in ([FM00], Section 6) and Theorem (1.3).

4.2. **Remark.** The main theorem of [VZ07] says that for a smooth variety X of n arbitrary dimension of Kodaira dimension 2 and assuming that the generic fiber of the Iitaka fibration is a smooth variety F, there is a constant N depending on the middle Betti number b_m of F and the index of F, such that $|NK_X|$ gives the Iitaka fibration. From (3.1) and (1.3), we prove that, the analogous bounds for the Betti number and the index, which only depends on the DCC set $\mathcal{A}$, exist even in the log case, provided the generic fiber is of dimension 2.

4.3. **Theorem.** *If (W, D) is a klt surface, L is a nef $\mathbb{Q}$-divisor (not necessarily effective) such that and $K_W + D + L$ is big. Assume that a is a positive integer such that aL is a Cartier divisor, the coefficients of D are in a DCC set $\mathcal{B}$, then we have a uniform $N = N(a, \mathcal{B})$ such that $|\lfloor N(K_W + D + L) \rfloor|$ gives a birational map.*

Proof of (4.1): We apply (4.3) to $W = C$, $D = B$, $L = M$. To check the assumptions, the coefficients of B are of the form

$$\frac{b + n - 1}{n}, \text{ for some } b \in \mathcal{A} \cup \{0\} \text{ and } n \in \mathbb{Z}_{>0},$$

(cf. [Ko07], Theorem 8.3.7(2)) which forms a DCC set depending on $\mathcal{A}$. And for a, it can be chosen as in the last part of the proof of (1.3). $\square$

Proof of (4.3): This is essentially proved in [VZ07] so here we only give an outline of the proof following [AM04]. As before, we can start by assuming that W is smooth. The main observation is that in fact many of the results in [AM04] can be strengthened in the way that instead of assuming $(X, B = \sum b_j B_j)$ $(b_j \in \mathcal{B})$ is big, we can assume that $(X, B + L)$ is big, where L is a nef line bundle such $L \cdot C$ is in another DCC set $\mathcal{C}$, then we get the same conclusion by changing all our constants $c = c(\mathcal{B})$ to $c = c(\mathcal{B}, \mathcal{C})$. In particular, we have the following:

4.4. **Proposition.** *Let the notation be as in (4.3). Then there is a uniform $\beta = \beta(\mathcal{B}, a)$ such that $K_X + \sum(1 - x_j) b_j B_j + L$ is big, provided $x_j \leq \beta$.*

Because of (4.4) and the fact that all positive b_j have a lower bound, we can indeed assume that all b_j are of the form n_j/m for some $m = m(\mathcal{B}, a)$.

Then the usual argument of cutting log canonical centers (cf. [To08]) works as long as we can prove that the volume of $K_X + \sum(1 - x_j) b_j B_j + L$ has a uniform lower bound.

Now we run the minimal model program, $f : (W, D) \to (W', D' = f_*(D))$, $L' = f_* L$, because $f^*(L') \geq L$, we have

$$H^0(W, \lfloor n(K_W + D + L) \rfloor) = H^0(W', \lfloor n(K_{W'} + D' + L') \rfloor), \forall n.$$

Case (1): $K_W + D$ is pseudo-effective, then we end with a minimal model (W', D') with $K_{W'} + D'$ is nef. If $K_{W'} + D'$ is big, then the Proposition follows from [AM04]. Otherwise,

$$(K_{W'} + D' + L)^2 = (K_{W'} + D') \cdot L' + L'^2 > 0,$$

which has a uniform lower bound by our assumption.

If $K_W + D$ is not pseudo-effective then we define the *pseudo-effective threshold*, which is the smallest number e such that $K_W + D + eL$ is pseudo-effective. Case (2): we end with a log del Pezzo surface (W', D') of Picard number 1. So

$$K_{W'} + D' + eL' \equiv 0.$$

The above discussion about the generalization of [AM04] indeed implies that $1 - e$ has a uniform lower bound. So $(K_{W'} + D' + L')^2 = (1 - e)L'^2$.

Case (3): we end with (W', D') with a Fano contraction to a curve. By the above argument we can assume that L' is not big. Then W' is of Picard group of rank 2 generated by the fiber of the Fano contraction and L'. Taking the intersection of $K'_W + D' + eL'$ with the fiber, we conclude that e is uniformly far from 1. Since the coefficients of D' have bounded denominators, the fact that $(K_{W'} + D') \cdot L'$ is positive actually implies it is uniformly away from 0. So the volume

$$(K_{W'} + D' + L')^2 = (1-e)(K_{W'} + D') \cdot L'$$

is bounded from below.

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